Local non-planarity of three dimensional surfaces for an invertible reconstruction: k-cuspal cells

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Abstract. This paper addresses the problem of the maximal recognition of hyperplanes for an invertible reconstruction of 3D discrete objects. k-cuspal cells are introduced as a three dimensional extension of discrete cusps defined by R.Breton. With k-cuspal cells local non planarity on discrete surfaces can be identified in a very straightforward way.

1 Introduction

For some years now, the discrete geometry community tries to propose an invertible reconstruction of a discrete object. A reconstruction is a transformation from the discrete to the Euclidean world that transforms a discrete object into an Euclidean one. The aim is to propose a reconstruction that is invertible (the discretisation of the reconstruction is equal to the original object), and that generates Euclidean objects with as few polygons as possible. Ideally for instance, the discretisation of a cube should be reconstructed as an Euclidean cube with only 6 faces. That is something that is not possible with a Marching Cube reconstruction that yields a number of polygons proportional to the number of discrete points. This is something we would like to avoid when handling very big discrete objects or multi-scale discrete objects. We consider the discrete analytical framework where the reconstruction is divided into two steps: In the first step, the boundary of a 3D discrete object is decomposed into discrete plane pieces [Rev91] [And03] (discrete straight lines segments in 2D) and in a second step those plane pieces are replaced by Euclidean polygons.

This approach has provided very good results in 2D especially with approaches based on parametric spaces such as the J. Vittone approach [VC00] [VC99]. The algorithm was adapted by R.Breton [BSDA03] to the standard model [And03] and generalized by M. Dexet to the upper dimensions [DA08] using the topological framework of abstract cells complexes of V. Kovalevsky [Kov93]. Reconstruction in 3D is however not very convincing so far. As we can see in Fig.1(b), an invertible reconstruction does not guarantee a natural reconstruction. When performing the recognition step, the biggest recognised piece of discrete plane may actually be too big. It does not take into account the local differential behaviour of the discrete surface. to avoid this problem, we introduce, in this paper, k-cuspal cells as the three dimensional extension of the 2D discrete

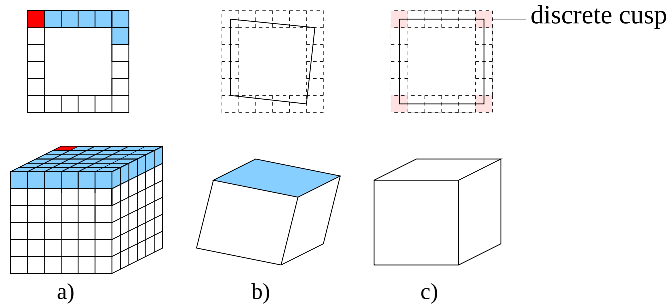

Fig. 1. Reconstructions examples based on maximal recognition. *a*) the discrete object, the colored pixels is the first set recognized like a hyperplane (a line). *b*) the reconstructed object. *c*) the expected result.

cusps [BSDA03]. A k-cuspal cell is a cell of dimension k on the boundary of a discrete object seen as a discrete abstract cell complex. The characterisation of a cell as a cuspal cell is very easy and can be performed on the fly during the recognition process. The aim to guide the recognition step in order to obtain a more *natural* reconstruction as illustrated in Fig.1(c). We start, in section 2 with some basics on discrete geometry and a recall on 2D discrete cusps. In section 3, we introduce k-cuspal cells which are the 3D extension of the 2D cusp points. We end section 3, with several illustrations of k-cuspal cells on a set of various discrete objects. We conclude in section 5 and discuss several perspectives.

2 Preliminaries

2.1 Basic Notations

Let $\mathbb{Z}^n$ be the subset of the nD euclidean space $\mathbb{R}^n$ that consists of all the integer coordinate points. A *discrete* (resp. euclidean) *point* is an element of $\mathbb{Z}^n$ (resp. $\mathbb{R}^n$). A *discrete* (resp. euclidean) *object* is a set of discrete (resp. euclidean) points.
We denote p_i the i-th coordinate of a point or vector p. The *voxel* $\mathbb{V}(p) \subset \mathbb{R}^n$ of a discrete nD point p is defined by $\mathbb{V}(p) = [p_1 - \frac{1}{2}, p_1 + \frac{1}{2}] \times ... \times [p_n - \frac{1}{2}, p_n + \frac{1}{2}]$. Two discrete points p and q in dimension n are *k-neighbours*, with $O \leq k \leq n$, if $|p_i - q_i|$ for $1 \leq i \leq n$, and $k \leq n - \sum_i^{n=1} |p_i - q_i|$.

An *abstract cell complex* [Kov93], $C = (E, B, dim)$, is a set E of abstract elements provided with an antisymetric, irreflexive, and transitive binary relation $B \subset E \times E$ called the *bounding relation*, and with a dimension function $dim : E \to I$ from E into the set I of non-negative integers such that $dim(e') < dim(e'')$ for all pairs $(e', e'') \in B$. A k dimensional cell is called a *k-cell*.

In a three dimensional discrete space, we describe an object by its abstract cell complex boundary. The boundary is a set of 0-cells, 1-cells and 2-cells (See Fig.2). In classical topology, the *Jordan theorem* states that every non-self-intersecting

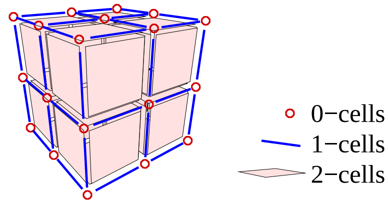

Fig. 2. An example of three dimensional abstract cells complexes

boundary in the euclidean space divides the space into an "inside" and an "outside". This theorem is easily verified for abstract cells complexes but with a lot of difficulties in classical discrete spaces. We know, furthermore, that if a discrete object is a 2-connected set of discrete points then its boundary can be described as a 2-dimensional manifold [Fra95]. In practice, this is very important in three dimensional segmentation when handling several objects in an image. For instance, in medical imaging, when dealing with several organs, working within the framework of abstract cell complexes allows to have a common 2-dimensional boundary between different organs. When reconstructing each organ we can ensure that the common boundary is identical for both reconstructions (see Fig.3).

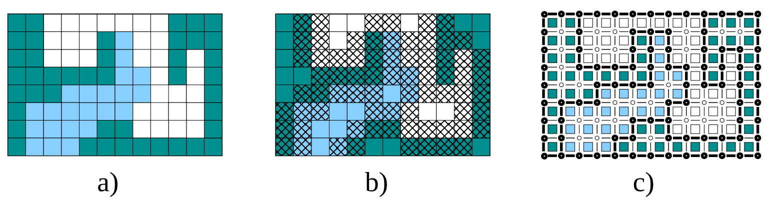

Fig. 3. a) Two dimensional example with several objects b) Example of classical boundary representation (pixel with grid are boundary pixels for its respective region) c) Boundary representation with abstract cells complexes.

A *standard discrete hyperplane* [And03] of dimension n and normal vector $C = (c_0, ..., c_n) \in \mathbb{R}^{n+1}$ is defined as the set of lattice points $p = (p_1, ..., p_n) \in \mathbb{Z}^n$ such that $-\frac{\sum_{i=1}^{n}(|c_i|)}{2} \leq c_0 + \sum_{i=1}^{n}(c_i p_i) < \frac{\sum_{i=1}^{n}(|c_i|)}{2}$ where $c_1 \geq 0$, or $c_1 = 0$ and $c_2 \geq 0$, or ..., or $c_1 = c_2 = ... = c_{n-1} = 0$ and $c_n \geq 0$. Standard hyperplanes are a particular case of *analytical discrete hyperplanes* defined by J.-P. Reveilles [Rev91]. The reason why standard discrete hyperplanes are interesting is because they are $(n-1)$-connected objects. The abstract cell complex boundary of a discrete 2-connected object can be seen as a set of standard hyperplane pieces as long as you shift the grid and take 0-cells as integer coordinate points.

2.2 A two dimensional solution to the maximal recognition problem: Discrete cusp [BSDA03]

In a two dimensional Euclidean space, a cusp is a point of a simple connected curve that does not have a tangent but that has a tangent on either side of the point (a "left" and a "right" tangent that are different). In the discrete world, everything is linear in the sense that if you have a curve then you can always decompose your curve into line segments. In [V96], A. Vialard has defined a discrete tangent on a point p of a 1-connected simple curve as the longest discrete line segment centered on p and included in the curve. It is easy to see that every point (except for the end points of an open curve), is always center of a discrete 1-connected line segment of at least length 3. This is not the case anymore if we consider 5 consecutive curve points. However, in [BSDA03], R.Breton, defined as discrete cusp, a discrete points that does not have a Vialard tangent of length at least 5.

Definition 1 (Discrete Cusp [BSDA03]). *A discrete point p in a discrete (locally) simple curve $C = \left\{p^1, p^2, \ldots, p^{i-2}, p^{i-1}, p, p^{i+1}, p^{i+2}, \ldots, p^{k-1}, p^k\right\}$ is a discrete cusp if the set $\left\{p^{i-2}, p^{i-1}, p, p^{i+1}, p^{i+2}\right\}$ is not a discrete straight line segment.*

Discrete cusps are particularly interesting when doing discrete analytical reconstruction. Discrete analytical 2D boundary reconstruction is performed in two steps: decompose the boundary into discrete line segments and replace each discrete line segment by a Euclidean line segment. By definition, if a discrete line segment $L = \left\{p^1, p^2 \ldots, p^{k-1}, p^k\right\}$ contains a discrete cusp p then p is one of the following points p_1, p_2, p_{k-1} or p_k. In Fig. 1(a), we see an example with the 2D square. Each of the corners of the square are discrete cusps. If we start a line recognition on the upper left corner, going clockwise, the longest possible recognized 1-connected line ends one point after the next corner (cusp). If we want the result of Fig. 1(c), the recognition needs to stop on the cusp and not one point after the cusp. Tracking down discrete cusps is here very useful since it allows us to guide the discrete analytical recognition step. Usually, a discrete cusp will lead to a vertex in the resulting reconstructed shape [BSDA03]. It is defined as a local discontinuity in the tangents and can actually be easily detected using the Freeman code [Free70] (see Fig.4).

3 A solution to the three dimensional maximal recognition problem: *k*-cuspal cells

3.1 Definitions

We are now going to discuss the extension of the notion of 2D discrete cusps into 3D. The main idea behind the definition of discrete cusps on three dimensional surfaces is that orthogonal projections of a three dimensional discrete standard line are two dimensional discrete standard lines [And03]. The three dimensional

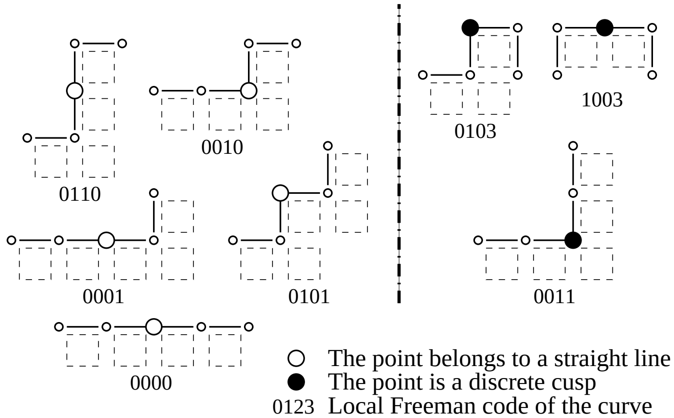

Fig. 4. Possible configurations of at most five discrete points with their freeman code. On the left points that aren't discrete cusps and on the right, configurations corresponding to discrete cusps.

extension of discrete cusps is found in all the orthogonal sections of the discrete object boundary. We are interested in the orthogonal sections crossing the voxels by their center. The k-cells, $k \in \{0, 1\}$, of the two dimensional space obtained by this kind of orthogonal sections represent $(k+1)$-cells in the three dimensional space.

Let us now define three-dimensional k-cuspel cells. Contrary to the dimension two, there are different types (dimensions) of cusps in dimension 3. In dimension two, only points (0-cells in the discrete cell complex representation) can be discrete cusps. In dimension three, we have 0-cuspal cells that are discrete points (0-cells) and 1-cuspal cells formed by a pair of discrete points (1-cells).

Let us first define 1-cuspal cells. When performing an orthogonal section through the center of voxels, a three dimensional 1-cuspal cell corresponds to a discrete 2D cusp found in the orthogonal section (see Fig.5).

Definition 2 (1-cuspal cells). *A 1-cell (in an abstract cell complex) is a 1-cuspal cell if and only if it belongs to the boundary of a discrete object and if its orthogonal section is a two dimensional discrete cusp.*

There are three kind of 1-cuspal cells since there are three ways (directions) of performing an orthogonal section. For example, if c^0 is a discrete cusp in the section $X = \alpha$ ($\alpha \in \mathbb{Z}$) then his associated 1-cell c^1 is said to be a 1-cuspal cell according to X.

Theorem 1. *Let $\mathcal{O} \subset \mathbb{Z}^3$ be a discrete object and $\mathcal{F}$ its boundary in an abstract cell complex representation. If $\mathcal{P} \subset \mathcal{F}$ is a standard hyperplane as abstract cell complex, then $\mathcal{P}$ does not contain any 1-cuspal cell.*

Proof. Let $\mathcal{P}$ be a discrete standard hyperplane of inequation $-w \le a_0 + a_1x + a_2y + a_3z < w$ (with $w = \frac{a_1+a_2+a_3}{2}$). The intersection of such hyperplane with

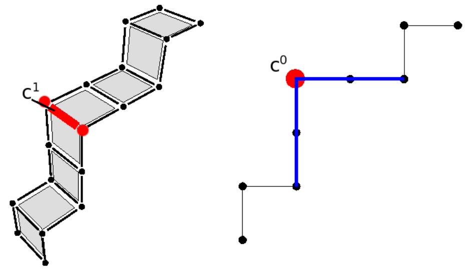

Fig. 5. A 1-cuspal cell c^1 correspond to a discrete cusp c^0 in an orthogonal section

an orthogonal section is a discrete two dimensional line $\mathcal{D}$ of inequation: $-w \leq a'_0 + a'_1x' + a'_2y' < w$ where $w \geq \frac{a'_1+a'_2}{2}$. That is the inequation of a thick line in $\mathbb{Z}^2$. Therefore, no discrete cusp is found in $\mathcal{D}$ and so, no 1-cuspal cell found in $\mathcal{P}$. □

A 1-cuspal cell can be seen as a 1-cell that belongs to two different hyperplanes. Each 1-cell is bounded by two 0-cells. We can see on Fig.5 that the two 0-cells bounding c^1 have the same cuspal property as c^0. Therefore the cuspal property of a 1-cell is inherited by its border.
Now that we have defined 1-cuspal cells, we can define 0-cuspal cells:

Definition 3 (0-cuspal cells). *A* 0*-cell of a discrete object boundary (represented by an abstract cell complex) that bounds three* 1*-cuspal cells according to the three different directions is a* 0*-cuspal cell (see Fig.6).*

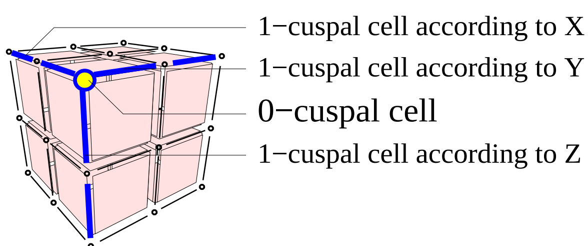

Fig. 6. Exemple of 0-cuspal cell.

Equivalently to what happens in dimension two, k-cuspal cells indicate 0-cells and 1-cells that can not be in the center of a somewhat large disk (orthogonal section of length at least 5 in the direction of the cusp) that is included in a standard analytical discrete plane. In Fig. 1, we see an example of a 3D cube. Each corner of the cube are O-cuspal cells and each edge of the cube are 1-cuspal cells. When performing three-dimensional reconstruction, there is, as first step, a discrete plane recognition step. The boundary of the discrete object is decomposed into discrete analytical standard 3D planes. Just as in dimension 2, 1-cuspal cells form good markers that allow us to stop the recognition process. Ideally, 1-cuspal cells and 0-cuspal cells should be on the border of the discrete analytical standard plane segments that are recognized.

3.2 Implementation and Results

A test platform was developed to represent discrete objects and their k-cuspal cells [ABL01]. The 1-cuspal cell detection is processed in linear time according to the number of 1-cells of the object boundary.

k-cuspal cells on discrete cubes: Firstly, we checked the localisation of k-cuspal cells on discrete cubes with various orientations in space. By construction, on the un-rotated cube, all edges are identified by 1-cuspal cells and vertices by 0-cuspal cells. We have then tried to find k-cuspal cells on several rotated cubes (see Fig.7). The rotation we used is the E. Andres discrete rotation [JA95].

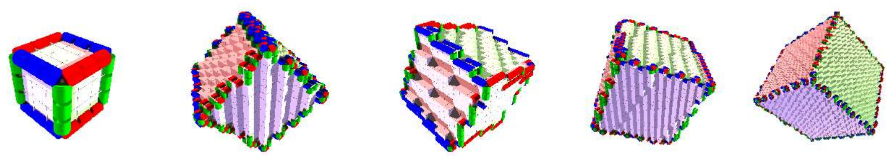

Fig. 7. 1-cuspal cells detection on rotated cubes.

All the 1-cuspal cells detected are on the rotated edges of the cube. As expected according theorem 1 no 1-cuspal cells has been detected on the faces of the cube. Each face of the cube is indeed a standard discrete plane. Depending on the orientation of the cube, the location of 0-cuspal can vary and do not always correspond to the location of the expected vertices of the cube. However, we notice that they are regularly distributed among the cube edges. This can be explained by the fact that we use a discrete rotations and because our window of convolution (of sort) is rather small.

k-cuspal cells on discrete spheres: We would expect no cuspal cell on discrete spheres. This is of course not true in general and depends highly on the size of the discrete sphere. Here, 1-cuspal cells represent local points where the curve can not be a standard hyperplane and may be an object edge. Looking for discrete spheres 1-cuspal cells localisation is interesting because spheres are locally tangent to a plane and have no edges (see Fig.8).

We apply our test on the E. Andres discrete analytical sphere model [And94]. On little spheres, a lot of k-cuspal cells are detected which was to be expected.

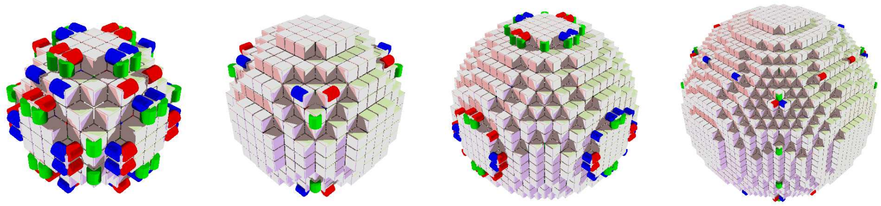

Fig. 8. Exemples of discrete sphere 1-cuspal cells detection.

When the radius grows, only a few k-cuspal cells are still detected. Usually this happens at specific discrete change of quadrant (there are 48 quadrants in 3D that are the equivalent to the 8 octants in 2D).

Random Polyhedrons: When testing with the discretisation of random polyhedrons, we can see in Fig.9 that not all discrete edges are composed of our 1-cuspal cells. This was of course to be expected since the length of the line segments we are testing are rather small (length of 5).

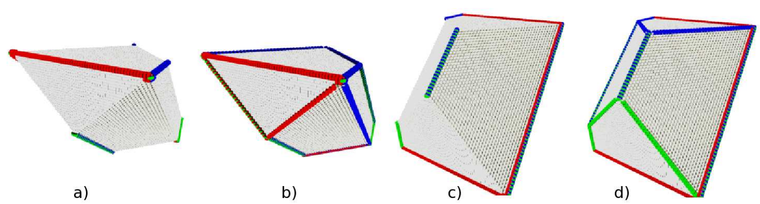

Fig. 9. k-cuspal cells detection on a random polyedron. a) and c) the detection is done with a window of size five. b) and d) the detection is done with a window of size seven

The faces of the discrete polyhedrons are discrete planes. It is rather easy to expend our definition for 1-cuspal cells by increasing the window by which we do our test (the length of the line segment we consider no define a 1-cuspal cell. This allows of course to detect smaller variations in the orientations of the faces (in the tangents on the boundary).

Stanford Bunny: The last series of tests was done on the Standford Bunny. The results are quite interesting considering the simplicity of the technique. We can see on the Fig.11, that even on a complex object like that, 1-cuspal cells detect many of the orientation changes on such a discrete object.

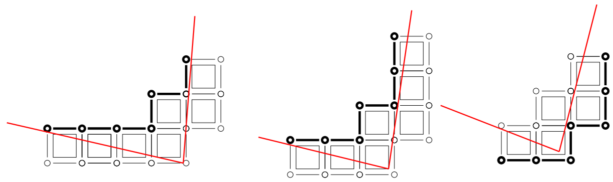

Fig. 10. Exemples of acute angles which are not detected with a window size lower than seven.

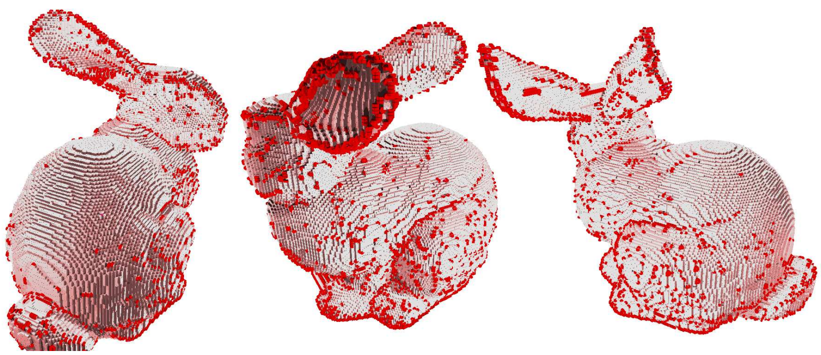

Fig. 11. k-cuspal cells detection on the stanford bunny

4 Conclusion

In this paper we introduced the notion of k-cuspal cells that are the extension to 3D of 2D discrete cusps introduced by R. Breton [BSDA03]. k-cuspal cells locate local non-planarity on three dimensional discrete surfaces. The k-cuspal cells improve the maximal recognition problem by allowing a more natural decomposition of a discrete surface into discrete plane pieces.

We have added the detection of k-cuspal cells in our multi-representation modeling software SpaMod [ABL01]. We hope to be able to propose now a more *natural* analytical reconstruction with help of these k-cuspal cells. This is one of the perspectives of this work.

The notion of k-cuspal cells has been developed so that it can be computed on the fly during the recognition process and so that it provides a very simple way of addressing planarity changes on discrete surfaces. This notion is enough when dealing with simply the recognition process but as soon as we are interested in a more detailed analysis of the non-planarity of discrete surfaces, a more extended notion of such cells seems necessary. Increasing the size of the neighborhood we are considering in order to characterize changes in planarity is one of the ongoing work.

We would like also to find a notion of *cuspal edge* that identifies a discrete 3D line segment as edge of two differently oriented discrete polygons. Right now the discrete cuspal cells we identify are not necessarily connected even though they are located on the edge.

Our analytical reconstruction is defined for any dimension, we can thus imagine a n dimensional extension of k-cuspal cells: k-cuspal cells could be recursively detected in the space orthogonal sections.

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